384×10^{-11} MeV/T

$g_e \mu_B \frac{\vec{S}_e}{\hbar}$

$\mu_N \frac{\vec{S}_p}{\hbar}$

245×10^{-14} MeV/T

1

ry

state

Hyperfine Splitting

The proton and electron both have a magnetic dipole moment.

$$\vec{\mu}_p = g_p \left(\frac{e}{2m_p}\right)\vec{S}_p \qquad \text{and} \qquad \vec{\mu}_e = -g_e\left(\frac{e}{2m_e}\right)\vec{S}_e$$

where $g_p = 5.59$ $\qquad g_e = 2\left(1 + \frac{\alpha}{2\pi} + \cdots\right)$

$g_e = 2.00231930436\,17(15)$

measured to 12 decimal places

From Classical Electrodynamics

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}\right] + \frac{2}{3}\mu_0\vec{\mu}\,\delta^3(\vec{r})$$

The B field due to the proton

$$H'_{hf} = -\underset{\text{electron}}{\vec{\mu}_e}\cdot\underset{\text{due to proton}}{\vec{B}}$$

$$H'_{hf} = \frac{\mu_0 g_p e^2}{8\pi m_p m_e}\,\frac{\left[3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e\right]}{r^3} + \frac{\mu_0 g_p e^2}{3 m_p m_e}\,\vec{S}_p\cdot\vec{S}_e\,\delta^3(\vec{r})$$

$$E'_{hf} = \langle \psi_{100} | H'_{hf} | \psi_{100}\rangle$$

For $\quad E'_{hf} = \frac{\mu_0 g_p e^2}{8\pi m_p m_e}\left\langle \frac{3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e}{r^3}\right\rangle + \frac{\mu_0 g_p e^2}{3 m_p m_e}\langle\vec{S}_p\cdot\vec{S}_e\rangle|\psi(0)|^2$

Problem 6.27: $\int(\vec{a}\cdot\hat{r})(\vec{b}\cdot\hat{r})\sin\theta\, d\theta\, d\phi = \frac{4}{3}\pi\,\vec{a}\cdot\vec{b}$

Hyperfine Splitting

Problem 6.27 continued.

$$\left\langle \frac{3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r}) - \vec{S}_p\cdot\vec{S}_e}{r^3} \right\rangle = \left\{ \int_0^\infty \frac{1}{r^3} |\psi(r)|^2 r^2 dr \right\} \left\{ \int\left[3(\vec{S}_p\cdot\hat{r})(\vec{S}_e\cdot\hat{r})\right]\sin\theta\, d\theta\, d\phi - \int \vec{S}_p\cdot\vec{S}_e \sin\theta\, d\theta\, d\phi \right\}$$

$$= \left\{ \int \frac{1}{r^3} |\psi(r)|^2 r^2 dr \right\} \left\{ 3\,\frac{4\pi}{3}\,\vec{S}_p\cdot\vec{S}_e - 4\pi\, \vec{S}_p\cdot\vec{S}_e \right\} = 0$$

So, all that remains is:

$$E^1_{hf} = \frac{\mu_0 g_p e^2}{3\pi m_p m_e a^3} \underbrace{\langle \vec{S}_p\cdot\vec{S}_e \rangle}_{\text{Spin-Spin Coupling}} \qquad \text{Recall: } \psi_{100}(r) = \frac{1}{\sqrt{\pi}}\frac{1}{a_0^{3/2}} e^{-r/a_0}$$

The "good" states are eigenvectors of the total spin: $\vec{S} = \vec{S}_p + \vec{S}_e$

$$\vec{S}_p \cdot \vec{S}_e = \frac{1}{2}\left(S^2 - S_e^2 - S_p^2 \right)$$

$$\langle \vec{S}_p\cdot\vec{S}_e \rangle = \frac{1}{2}\left(2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2 \right) = +\frac{1}{4}\hbar^2 \quad \text{(triplet)}$$

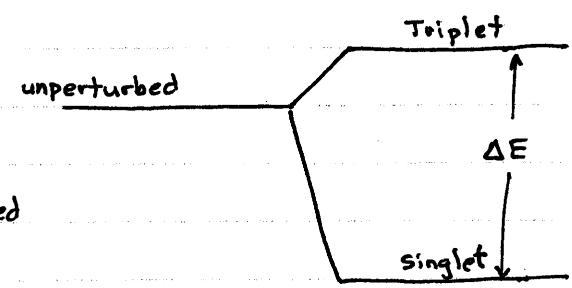
$$\langle \vec{S}_p\cdot\vec{S}_e \rangle = \frac{1}{2}\left(0\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2 \right) = -\frac{3}{4}\hbar^2 \quad \text{(singlet)}$$

So, now the hyperfine splitting can be written as:

$$E^1_{hf} = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4} \begin{cases} \left(+\frac{1}{4}\right) & \text{for the triplet state} \\ \left(-\frac{3}{4}\right) & \text{for the singlet state} \end{cases}$$

Spin-spin coupling breaks the degeneracy of the hydrogen ground state, ψ_{100}.

$$\Delta E = \frac{4 g_p \hbar^4}{3 m_p m_e^2 c^2 a^4} = 5.88 \times 10^{-6}\ eV$$



1. What is the wavelength of the emitted photon?

$$\Delta E = \frac{hc}{\lambda} \qquad \lambda = \frac{hc}{\Delta E} = \frac{1240\ eV\cdot nm}{5.88 \times 10^{-6}\ eV} \approx 2.11 \times 10^{8}\ nm = 0.21\ \text{meters}$$

$$\boxed{\lambda = 21\ cm}$$

2. What is the lifetime of the triplet state?

and Waves"

a.) The total power radiated by a magnetic dipole is:

$$P_m = \frac{c k^4 \mu_e^2}{3} \frac{\mu_0}{4\pi} \qquad \text{where } \mu_e = \frac{e\hbar}{2m_e} \quad \text{mag. dipole moment of the } e^-$$

The approximate lifetime of a state is the energy radiated per transition divided by the power radiated.

$$\tau \approx \frac{\Delta E}{P_M} = \frac{\hbar\omega}{P_M} = \hbar k c \frac{3}{c k^4 \mu_e^2 (\mu_0/4\pi)}$$

Lifetime of the triplet state $(\lambda = 21\,cm)$

$$\tau \simeq \frac{3\hbar}{k^3 \left(\frac{e\hbar}{2m_e}\right)^2 \left(\frac{\mu_0}{4\pi}\right)} = \frac{12\, m_e^2}{k^3 e^2 \hbar \left(\frac{\mu_0}{4\pi}\right)}$$

$$\tau \simeq \left(\frac{\lambda}{2\pi}\right)^3 \frac{12\, m_e^2}{e^2 \hbar \left(\frac{\mu_0}{4\pi}\right)} =$$

$$\tau \simeq \left(\frac{0.21\,m}{2\pi}\right)^3 \frac{12\,(9.11\times10^{-31}\,kg)^2}{(1.602\times10^{-19}\,C)^2(1.055\times10^{-34}\,J\cdot s)(10^{-7}\,N\cdot s^2/C^2)}$$

$$\tau \simeq 1.37\times10^{15}\,sec\left(\frac{1\,yr}{3.16\times10^7\,s}\right) = 4.35\times10^7\ years$$

$$\boxed{\tau \simeq 43\ \text{million years}}$$

5.7884×10^{-11} MeV/T

electron: $\vec{\mu}_e = -g_e\left(\frac{e}{2m_e}\right)\vec{S}_e = -g_e \underline{\underline{\mu_B}} \frac{\vec{S}_e}{\hbar}$

proton: $\vec{\mu}_p = g_p\left(\frac{e}{2m_p}\right)\vec{S}_p = g_p \underline{\underline{\mu_N}} \frac{\vec{S}_p}{\hbar}$

3.15245×10^{-14} MeV/T.

Deuteron: $\vec{\mu}_D = \underline{\underline{g_D}} \mu_N \frac{\vec{S}_D}{\hbar}$

0.857

Deuteron continued:

	g_D	
$S=1 \quad L=0 \quad (J=1)$	$\mu = 0.879$	Theory
$S=1 \quad L=2 \quad (J=1)$	$\mu = 0.310$	Theory

Deuteron is mostly in the $S=1 \quad L=0$ state

Magnetic Dipole of a neutron:

$\mu_{neutron} = g_{Neutron}\, \mu_N = -1.913\, \mu_N$